Arithmetic Ramsey Theory via ultrafilters and nonstandard methods

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Arithmetic Ramsey theory is an area of combinatorics that focuses on the existence, for any finite coloring (partition) of the natural numbers, of "monochromatic patterns" defined by arithmetic operations. More precisely, the typical problem has the following form: "Given a family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$, is it true that for every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ one can always find a "configuration" $F \in \mathcal{F}$ which is monochromatic, *i.e.* $F \subseteq C_j$ for some j?".

Plenty of "monochromatic configurations" have been proved to exist. Historically, the first examples are provided by Schur's triples $\{x, y, x+y\}$ (Schur's Theorem, 1918), and ℓ-term arithmetic progressions $\{a, a+d, \ldots, a+(\ell-1)d\}$ for any fixed ℓ (van der Waerden's Theorem, 1927). A fundamental result on infinite configurations is Hindman's Theorem, which states that one can always find an infinite sequence $(x_n \mid n \in \mathbb{N})$ such that all sums of its distinct elements $x_{n_1} + \ldots + x_{n_k}$ are monochromatic.

Despite intense research activity, there are still several "simple" unsolved problems in this field. Probably the most important one concerns the possibility of combining sums and products: "Is the pattern $x, y, x+y, xy$ monochromatic?"

Various techniques have been successfully applied to the problems of arithmetic Ramsey theory, including ergodic theory, discrete harmonic analysis, topological dynamics, algebra in the space of ultrafilters. In this seminar I will show some recent applications in this area of nonstandard analysis methods combined with the use of ultrafilters. A relevant example, where we combine sums and quotients in an infinitary form, is the following extension of Hindman's Theorem (joint work with R. Mennuni, L. Luperi Baglini, M. Ragosta, and A. Vegnuti):

- For every finite colouring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there exists an increasing sequence $(x_i \mid i \in \mathbb{N})$ such that, for all $k < \ell$ and all $i_1 < \ldots < i_k < \ell$, the following are monochromatic:
$$x_{i_1} + \ldots + x_{i_k},\ \frac{x_{i_{k+1}} + \ldots + x_{i_\ell}}{x_{i_1} + \ldots + x_{i_k}} \subseteq C_j.$$

In particular, this yields partition regularity of the pattern $x, y, x+y, \frac{y}{x}$ over $\mathbb{N}$.